A Comprehensive Explanation of Global Convergence in Modern Hopfield Networks

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1 Introduction

This article provides a comprehensive explanation of the global convergence theory of the Modern Hopfield Network [1], covering energy and stationary points. It also extends the discussion of the initial point $\boldsymbol{\xi}_0$ from a bounded subset in the original paper to any vector in $\mathbb{R}^d$.

2 Notations

1. The p-norm of a vector is denoted: $\left\|\cdot\right\|_p, 1 \leqslant p$

2. Let $\beta > 0, \beta \in \mathbb{R}$

3. Let $[N] = \{1, \cdots, N\}$

3 Softmax Function

$$\mathbf{p}\left(\mathbf{x}\right) = \text{softmax}\left(\beta\mathbf{x}\right)$$

$$p_i = \left[\text{softmax}\left(\beta\mathbf{x}\right)\right]_i = \frac{\exp\left(\beta x_i\right)}{\sum_{k=1}^{N}\exp\left(\beta x_k\right)} \tag{1}$$

where $\mathbf{x}_i \in \mathbb{R}^N$.

4 Log-Sum-Exp Function

$$\text{lse}\left(\beta, \mathbf{x}\right) = \beta^{-1}\ln\left(\sum_{k=1}^{N}\exp\left(\beta x_k\right)\right) \tag{2}$$

where $\mathbf{x}_i \in \mathbb{R}^N$.

5 Lemma 1 LSE Gradient

$$\nabla_{\mathbf{x}}\,\text{lse}\left(\beta, \mathbf{x}\right) = \text{softmax}\left(\beta\mathbf{x}\right) \tag{3}$$

where $\mathbf{x} \in \mathbb{R}^N$.

$$\nabla_{\boldsymbol{\xi}}\,\text{lse}\left(\beta, \boldsymbol{X}^T\boldsymbol{\xi}\right) = \boldsymbol{X}\,\text{softmax}\left(\beta\boldsymbol{X}^T\boldsymbol{\xi}\right) \tag{4}$$

where $\boldsymbol{X} \in \mathbb{R}^{d\times N}, \boldsymbol{\xi} \in \mathbb{R}^d$.

6 Lemma 2 LSE Hessian

The Jacobian $J_s\left(\mathbf{x}\right)$ of $\mathbf{p}\left(\mathbf{x}\right) = \text{softmax}\left(\beta\mathbf{x}\right)$ is

$$J_s\left(\mathbf{x}\right) = \frac{\partial\,\text{softmax}\left(\beta\mathbf{x}\right)}{\partial\mathbf{x}} = \beta\left(\text{diag}\left(\mathbf{p}\right) - \mathbf{p}\mathbf{p}^T\right) \tag{5}$$

where $\mathbf{x} \in \mathbb{R}^N$.
The Jacobian $J\left(\boldsymbol{\xi}\right)$ of $\boldsymbol{X}\mathbf{p}\left(\boldsymbol{X}^T\boldsymbol{\xi}\right) = \boldsymbol{X}\,\text{softmax}\left(\beta\boldsymbol{X}^T\boldsymbol{\xi}\right)$ is

$$J\left(\boldsymbol{\xi}\right) = \frac{\partial\left(\boldsymbol{X}\,\text{softmax}\left(\beta\boldsymbol{X}^T\boldsymbol{\xi}\right)\right)}{\partial\boldsymbol{\xi}} = \beta\boldsymbol{X}\left(\text{diag}\left(\mathbf{p}\right) - \mathbf{p}\mathbf{p}^T\right)\boldsymbol{X}^T = \boldsymbol{X}J_s\boldsymbol{X}^T \tag{6}$$

where $\boldsymbol{X} \in \mathbb{R}^{d\times N}, \boldsymbol{\xi} \in \mathbb{R}^d$.

7 Lemma 3 LSE Hessian is Symmetric and Positive Semi-Definite

The Jacobian $J_s(\mathbf{x})$ of $\mathbf{p}(\mathbf{x}) = \operatorname{softmax}(\beta\mathbf{x})$ is symmetric and positive semi-definite where $\mathbf{x} \in \mathbb{R}^N$.
The Jacobian $J(\boldsymbol{\xi})$ of $\boldsymbol{X}\mathbf{p}\left(\boldsymbol{X}^T\boldsymbol{\xi}\right) = \boldsymbol{X}\operatorname{softmax}\left(\beta\boldsymbol{X}^T\boldsymbol{\xi}\right)$ is symmetric and positive semi-definite where $\boldsymbol{X} \in \mathbb{R}^{d\times N}, \boldsymbol{\xi} \in \mathbb{R}^d$.

Proof. According to 5, 6, they are symmetric. Given any $\mathbf{x} \in \mathbb{R}^N$, we have

$$\mathbf{x}^T J_s \mathbf{x} = \beta\mathbf{x}^T\left(\operatorname{diag}(\mathbf{p}) - \mathbf{p}\mathbf{p}^T\right)\mathbf{x} = \beta\left[\sum_i p_i x_i^2 - \left(\sum_i p_i x_i\right)^2\right] \geqslant 0 \quad (7)$$

Given any $\boldsymbol{\xi} \in \mathbb{R}^d$, we have

$$\boldsymbol{\xi}^T J\boldsymbol{\xi} = \boldsymbol{\xi}^T \boldsymbol{X} J_s \boldsymbol{X}^T\boldsymbol{\xi} = \left(\boldsymbol{X}^T\boldsymbol{\xi}\right)^T J_s\left(\boldsymbol{X}^T\boldsymbol{\xi}\right) \geqslant 0 \quad (8)$$

Hence they are positive semi-definite.

$\square$

8 Lemma 4 LSE is Convex

$\operatorname{lse}(\beta, \mathbf{x})$ is convex with respect to $\mathbf{x}$ where $\mathbf{x} \in \mathbb{R}^N$.
$\operatorname{lse}\left(\beta, \boldsymbol{X}^T\boldsymbol{\xi}\right)$ is convex with respect to $\boldsymbol{\xi}$ where $\boldsymbol{X} \in \mathbb{R}^{d\times N}, \boldsymbol{\xi} \in \mathbb{R}^d$.

Proof. The Jacobian of the $\operatorname{softmax}$ is Hessian of the lse according to 3, 4. The Jacobian of the $\operatorname{softmax}$ is positive semi-definite according to Lemma 3. Therefore lse is convex.

$\square$

9 Definitions

9.1 Point-to-set Map

A point-to-set map $\mathcal{A}$ from a set X into a set Y is defined as

$$\mathcal{A} : X \to \mathscr{P}(Y) \tag{9}$$

which assigns a subset of Y to each point of X, where $\mathscr{P}(Y)$ denotes the power set of Y.

9.2 Closed

Suppose X and Y are two topological spaces. A point-to-set map $\mathcal{A} : X \to \mathscr{P}(Y)$ is said to be closed at $\mathbf{x}_0 \in X$ if:

$$\left. \begin{array}{l} \mathbf{x}_t \to \mathbf{x}_0 \text{ as } t \to \infty \\ \mathbf{y}_t \to \mathbf{y}_0 \text{ as } t \to \infty \\ \mathbf{x}_t \in X \\ \mathbf{y}_t \in \mathcal{A}(\mathbf{x}_t) \end{array} \right\} \implies \mathbf{y}_0 \in \mathcal{A}(\mathbf{x}_0) \tag{10}$$

A point-to-set map $\mathcal{A}$ is said to be closed on $S \subset X$ if it is closed at every point of S.

9.3 Fixed Point

A fixed point of a point-to-set map

$$\mathcal{A} : X \to \mathscr{P}(X) \tag{11}$$

is any point $\mathbf{x} \in X$ for which

$$\mathcal{A}(\mathbf{x}) = \{\mathbf{x}\} \tag{12}$$

A generalized fixed point of $\mathcal{A}$ is any point $\mathbf{x} \in X$ for which

$$\mathbf{x} \in \mathcal{A}(\mathbf{x}) \tag{13}$$

9.4 Uniformly Compact

A point-to-set map $\mathcal{A} : X \to \mathscr{P}(Y)$ is said to be uniformly compact on X if there exists a compact set $H \subset Y$ independent of $\mathbf{x}$ such that:

$$\mathcal{A}(\mathbf{x}) \subset H, \forall \mathbf{x} \in X \tag{14}$$

9.5 Monotonic

Let

$$\mathrm{E} : X \to \mathbb{R} \tag{15}$$

be a continuous function. The map

$$\mathcal{A} : X \to \mathscr{P}(X) \tag{16}$$

is said to be monotonic with respect to E at $\mathbf{x} \in X$ if

$$\forall \mathbf{y} \in \mathcal{A}(\mathbf{x}), \mathrm{E}(\mathbf{y}) \leqslant \mathrm{E}(\mathbf{x}) \tag{17}$$

$\mathcal{A}$ is said to be strictly monotonic with respect to E at $\mathbf{x} \in X$ if

$$\mathcal{A}(\mathbf{x}) \neq \{\mathbf{x}\} \implies \forall \mathbf{y} \in \mathcal{A}(\mathbf{x}), \mathrm{E}(\mathbf{y}) < \mathrm{E}(\mathbf{x}) \tag{18}$$

9.6 Global Convergence

Let X be a set and $\boldsymbol{\xi}_0 \in X$ a given point. Then an algorithm, $\mathcal{A}$, with initial point $\boldsymbol{\xi}_0$ is a point-to-set map $\mathcal{A} : X \to \mathscr{P}(X)$ which generates a sequence $\{\boldsymbol{\xi}_t\}_{t=0}^{\infty}$ via the rule

$$\boldsymbol{\xi}_{t+1} \in \mathcal{A}(\boldsymbol{\xi}_t), t = 0, 1, \cdots \tag{19}$$

$\mathcal{A}$ is said to be globally convergent if: Given any chosen initial point $\boldsymbol{\xi}_0$, the sequence $\{\boldsymbol{\xi}_t\}_{t=0}^{\infty}$ generated by $\boldsymbol{\xi}_{t+1} \in \mathcal{A}(\boldsymbol{\xi}_t), t = 0, 1, \cdots$ (or a subsequence) converges to a point for which a necessary condition of optimality holds.

10 Modern Hopfield Network

We have patterns that are represented by the matrix:

$$\boldsymbol{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_N) \tag{20}$$

where $\mathbf{x}_i \in \mathbb{R}^d$. Thus $\boldsymbol{X} \in \mathbb{R}^{d \times N}$. The largest norm of a pattern is:

$$M = \max_{i \in [N]} \|\mathbf{x}_i\|_2 \tag{21}$$

The query or state of the Hopfield Network is $\boldsymbol{\xi} \in \mathbb{R}^d$. Define energy $\mathrm{E}\left(\boldsymbol{\xi}\right)$ for a continuous query or state $\boldsymbol{\xi}$:

$$\mathrm{E}\left(\boldsymbol{\xi}\right) = -\operatorname{lse}\left(\beta, \boldsymbol{X}^T \boldsymbol{\xi}\right) + \frac{1}{2}\boldsymbol{\xi}^T \boldsymbol{\xi} + \beta^{-1}\ln N + \frac{1}{2}M^2 \tag{22}$$

11 New Update Rule

$$\boldsymbol{\xi}_{t+1} = \mathcal{F}\left(\boldsymbol{\xi}_t\right) = \boldsymbol{X}\mathbf{p} = \boldsymbol{X}\operatorname{softmax}\left(\beta \boldsymbol{X}^T \boldsymbol{\xi}_t\right) \tag{23}$$

where

$$\mathbf{p} = \operatorname{softmax}\left(\beta \boldsymbol{X}^T \boldsymbol{\xi}_t\right) \tag{24}$$

12 Lemma 5 CCCP Algorithm

Consider an energy function $\mathrm{E}: X \to \mathbb{R}$:

$$\mathrm{E}\left(\boldsymbol{\xi}\right) = \mathrm{E}_1\left(\boldsymbol{\xi}\right) - \mathrm{E}_2\left(\boldsymbol{\xi}\right) \tag{25}$$

where E_1 and E_2 are differentiable convex functions of $\boldsymbol{\xi}$ respectively and X is a convex set. Then the discrete CCCP (Concave Convex Procedure Algorithm) algorithm [2]

$$\boldsymbol{\xi}_{t+1} \in \mathcal{A}\left(\boldsymbol{\xi}_t\right), t = 0, 1, \cdots \tag{26}$$

given by:

$$\nabla_{\boldsymbol{\xi}}\mathrm{E}_1\left(\boldsymbol{\xi}_{t+1}\right) = \nabla_{\boldsymbol{\xi}}\mathrm{E}_2\left(\boldsymbol{\xi}_t\right) \tag{27}$$

guarantees $\mathcal{A}$ to be monotonic with respect to E.
In addition, if E_1 or E_2 is strictly convex function, and $\boldsymbol{\xi}_{t+1} \neq \boldsymbol{\xi}_t$, then

$$\mathrm{E}\left(\boldsymbol{\xi}_{t+1}\right) < \mathrm{E}\left(\boldsymbol{\xi}_t\right) \tag{28}$$

Proof. E_1 and E_2 are convex functions, so that $\forall t \geqslant 0$,

$$\left.\begin{array}{l} \mathrm{E}_1\left(\boldsymbol{\xi}_t\right) \geqslant \mathrm{E}_1\left(\boldsymbol{\xi}_{t+1}\right) + \left(\nabla \mathrm{E}_1\left(\boldsymbol{\xi}_{t+1}\right)\right)^T \left(\boldsymbol{\xi}_t - \boldsymbol{\xi}_{t+1}\right) \\ \mathrm{E}_2\left(\boldsymbol{\xi}_{t+1}\right) \geqslant \mathrm{E}_2\left(\boldsymbol{\xi}_t\right) + \left(\nabla \mathrm{E}_2\left(\boldsymbol{\xi}_t\right)\right)^T \left(\boldsymbol{\xi}_{t+1} - \boldsymbol{\xi}_t\right) \\ \nabla \mathrm{E}_1\left(\boldsymbol{\xi}_{t+1}\right) = \nabla \mathrm{E}_2\left(\boldsymbol{\xi}_t\right) \end{array}\right\}$$

$$\implies \mathrm{E}_1\left(\boldsymbol{\xi}_t\right) + \mathrm{E}_2\left(\boldsymbol{\xi}_{t+1}\right) \geqslant \mathrm{E}_1\left(\boldsymbol{\xi}_{t+1}\right) + \mathrm{E}_2\left(\boldsymbol{\xi}_t\right)$$
$$\implies \mathrm{E}_1\left(\boldsymbol{\xi}_{t+1}\right) - \mathrm{E}_2\left(\boldsymbol{\xi}_{t+1}\right) \leqslant \mathrm{E}_1\left(\boldsymbol{\xi}_t\right) - \mathrm{E}_2\left(\boldsymbol{\xi}_t\right)$$
$$\implies \mathrm{E}\left(\boldsymbol{\xi}_{t+1}\right) \leqslant \mathrm{E}\left(\boldsymbol{\xi}_t\right) \tag{29}$$

Therefore $\mathcal{A}$ is monotonic with respect to E.
In addition, if E_1 is strictly convex function, and $\boldsymbol{\xi}_{t+1} \neq \boldsymbol{\xi}_t$, then

$$\left.\begin{array}{l} \mathrm{E}_1\left(\boldsymbol{\xi}_t\right) > \mathrm{E}_1\left(\boldsymbol{\xi}_{t+1}\right) + \left(\nabla \mathrm{E}_1\left(\boldsymbol{\xi}_{t+1}\right)\right)^T \left(\boldsymbol{\xi}_t - \boldsymbol{\xi}_{t+1}\right) \\ \mathrm{E}_2\left(\boldsymbol{\xi}_{t+1}\right) \geqslant \mathrm{E}_2\left(\boldsymbol{\xi}_t\right) + \left(\nabla \mathrm{E}_2\left(\boldsymbol{\xi}_t\right)\right)^T \left(\boldsymbol{\xi}_{t+1} - \boldsymbol{\xi}_t\right) \\ \nabla \mathrm{E}_1\left(\boldsymbol{\xi}_{t+1}\right) = \nabla \mathrm{E}_2\left(\boldsymbol{\xi}_t\right) \end{array}\right\}$$

$$\implies \mathrm{E}_1\left(\boldsymbol{\xi}_t\right) + \mathrm{E}_2\left(\boldsymbol{\xi}_{t+1}\right) > \mathrm{E}_1\left(\boldsymbol{\xi}_{t+1}\right) + \mathrm{E}_2\left(\boldsymbol{\xi}_t\right)$$
$$\implies \mathrm{E}_1\left(\boldsymbol{\xi}_{t+1}\right) - \mathrm{E}_2\left(\boldsymbol{\xi}_{t+1}\right) < \mathrm{E}_1\left(\boldsymbol{\xi}_t\right) - \mathrm{E}_2\left(\boldsymbol{\xi}_t\right)$$
$$\implies \mathrm{E}\left(\boldsymbol{\xi}_{t+1}\right) < \mathrm{E}\left(\boldsymbol{\xi}_t\right) \tag{30}$$

$\square$

13 Lemma 6 Convex Concave Decomposition

The energy function $\mathrm{E}\left(\boldsymbol{\xi}\right)$ is the difference of a strictly convex function $\mathrm{E}_1\left(\boldsymbol{\xi}\right)$ and a convex function $\mathrm{E}_2\left(\boldsymbol{\xi}\right)$:

$$\mathrm{E}\left(\boldsymbol{\xi}\right) = \mathrm{E}_1\left(\boldsymbol{\xi}\right) - \mathrm{E}_2\left(\boldsymbol{\xi}\right)$$
$$\mathrm{E}_1\left(\boldsymbol{\xi}\right) = \frac{1}{2}\boldsymbol{\xi}^T\boldsymbol{\xi} + \beta^{-1}\ln N + \frac{1}{2}M^2 = \frac{1}{2}\boldsymbol{\xi}^T\boldsymbol{\xi} + C_1 \tag{31}$$
$$\mathrm{E}_2\left(\boldsymbol{\xi}\right) = \mathrm{lse}\left(\beta, \boldsymbol{X}^T\boldsymbol{\xi}\right)$$

where C_1 does not depend on $\boldsymbol{\xi}$.

Proof. $\frac{1}{2}\boldsymbol{\xi}^T\boldsymbol{\xi}$ is a strictly convex function.
According to Lemma 4, lse is a convex function.

$\square$

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14 Lemma 7 Constraint Set

Let

$$S = \{\boldsymbol{\xi} \| \|\boldsymbol{\xi}\|_2 \leqslant M\} \tag{32}$$

which is a convex and compact set.
Then

$$\forall t \geqslant 0, \boldsymbol{\xi}_{t+1} \in S \tag{33}$$

where $\boldsymbol{\xi}_{t+1}$ is from update rule 23.

Proof.

$$\left\|\boldsymbol{\xi}_{t+1}\right\|_2 = \|\boldsymbol{X}\mathbf{p}\|_2 = \left\|\sum_{i=1}^{N} p_i \mathbf{x}_i\right\|_2 \leqslant \sum_{i=1}^{N} p_i \|\mathbf{x}_i\|_2 \leqslant \sum_{i=1}^{N} p_i M = M \tag{34}$$

□

15 Lemma 8 Constraint Function

Define function

$$c(\boldsymbol{\xi}) = \boldsymbol{\xi}^T \boldsymbol{\xi} - M \tag{35}$$

Then

$$S = \{\boldsymbol{\xi} : c(\boldsymbol{\xi}) \leqslant 0\} \tag{36}$$

where S is defined by 32.

Proof.

$$\forall \boldsymbol{\xi}, c(\boldsymbol{\xi}) \leqslant 0 \iff \boldsymbol{\xi} \in S \tag{37}$$

□

16 Lemma 9 Convergence Theorem 1

Let a point-to-set map $\mathcal{A} : X \rightarrow \mathscr{P}(X)$ be a point-to-set map (an algorithm) that given a point $\boldsymbol{\xi}_0 \in X$ generates a sequence $\{\boldsymbol{\xi}_t\}_{t=0}^{\infty}$ through the iteration $\boldsymbol{\xi}_{t+1} \in \mathcal{A}(\boldsymbol{\xi}_t)$. Suppose

1. All points $\boldsymbol{\xi}_{t+1}, t \geqslant 0$ are in a compact set $\Omega \subset X$.

2. $\mathcal{A}$ is monotonic with respect to a continuous function E.

Then $\mathrm{E}(\boldsymbol{\xi}_t) \rightarrow \mathrm{E}(\boldsymbol{\xi}^*)$ for $t \rightarrow \infty$ and a limit point $\boldsymbol{\xi}^*$.

Proof. All points $\boldsymbol{\xi}_{t+1}, t \geqslant 0$ are in a compact set $\Omega \subset X$. Hence $\{\boldsymbol{\xi}_t\}_{t=0}^{\infty}$ must have a convergent subsequence

$$\boldsymbol{\xi}_{t_k} \rightarrow \boldsymbol{\xi}^*, \text{ as } k \rightarrow \infty \tag{38}$$

where $\boldsymbol{\xi}^* \in \Omega$ is a limit point. Continuity of E provides

$$\mathrm{E}\left(\boldsymbol{\xi}_{t_k}\right) \rightarrow \mathrm{E}\left(\boldsymbol{\xi}^*\right), \text{ as } k \rightarrow \infty \tag{39}$$

$\mathcal{A}$ is monotonic with respect to E. Hence

$$\forall t \geqslant 0, \mathrm{E}\left(\boldsymbol{\xi}_t\right) \geqslant \mathrm{E}\left(\boldsymbol{\xi}^*\right) \tag{40}$$

Using 39 and the definition of limit, given $\varepsilon_1 > 0$, there is a t_{ε_1} such that

$$\mathrm{E}\left(\boldsymbol{\xi}_{t_{\varepsilon_1}}\right) < \mathrm{E}\left(\boldsymbol{\xi}^*\right) + \varepsilon_1 \tag{41}$$

$\mathcal{A}$ is monotonic with respect to E, then

$$\forall t > t_{\varepsilon_1}, \mathrm{E}\left(\boldsymbol{\xi}_t\right) \leqslant \mathrm{E}\left(\boldsymbol{\xi}_{t_{\varepsilon_1}}\right) < \mathrm{E}\left(\boldsymbol{\xi}^*\right) + \varepsilon_1 \tag{42}$$

40, 42 then yield

$$\forall t > t_{\varepsilon_1}, \left|\mathrm{E}\left(\boldsymbol{\xi}_t\right) - \mathrm{E}\left(\boldsymbol{\xi}^*\right)\right| < \varepsilon_1 \tag{43}$$

Therefore

$$\mathrm{E}\left(\boldsymbol{\xi}_t\right) \rightarrow \mathrm{E}\left(\boldsymbol{\xi}^*\right), \text{ as } t \rightarrow \infty \tag{44}$$

$\square$

17 Theorem 1 Global Convergence: Energy

The update rule 23 converges globally: For $\boldsymbol{\xi}_{t+1} = \mathcal{F}(\boldsymbol{\xi}_t)$, the energy $\mathrm{E}(\boldsymbol{\xi}_t) \to \mathrm{E}(\boldsymbol{\xi}^*)$ for $t \to \infty$ and a limit point $\boldsymbol{\xi}^*$.

Proof. Define minimization problem:

$$\min_{\boldsymbol{\xi}} \mathrm{E}(\boldsymbol{\xi}) = \min_{\boldsymbol{\xi}} (\mathrm{E}_1(\boldsymbol{\xi}) - \mathrm{E}_2(\boldsymbol{\xi})) \tag{45}$$

where $\mathrm{E}_1, \mathrm{E}_2$ are defined by 31.

1. **Applying CCCP Algorithm**

   According to 27, let

   $$\nabla_{\boldsymbol{\xi}} \mathrm{E}_1(\boldsymbol{\xi}_{t+1}) = \nabla_{\boldsymbol{\xi}} \mathrm{E}_2(\boldsymbol{\xi}_t) \tag{46}$$

   Using 4, 31, the resulting update rule is:

   $$\boldsymbol{\xi}_{t+1} = \boldsymbol{X} \operatorname{softmax}\left(\beta \boldsymbol{X}^T \boldsymbol{\xi}_t\right) \tag{47}$$

   which is equivalent to 23.

   Let

   $$\mathcal{A}(\boldsymbol{\xi}_t) = \left\{ \boldsymbol{X} \operatorname{softmax}\left(\beta \boldsymbol{X}^T \boldsymbol{\xi}_t\right) \right\} \tag{48}$$

2. **Energy Convergence**

   Since

   (a) According to 32, 33, all points $\boldsymbol{\xi}_{t+1}, t \geqslant 0$ are in a compact set S.
   (b) According to Lemma 5, $\mathcal{A}$ is monotonic with respect to E.
   (c) Obviously E is continuous.

   then using Lemma 9

   $$\mathrm{E}(\boldsymbol{\xi}_t) \to \mathrm{E}(\boldsymbol{\xi}^*), \text{ as } t \to \infty \tag{49}$$

   $\square$

18 Lemma 10 MM Algorithm

Consider the optimization function $E : X \to \mathbb{R}$:

$$E(\boldsymbol{\xi}) = E_1(\boldsymbol{\xi}) - E_2(\boldsymbol{\xi}), \boldsymbol{\xi} \in \Omega \tag{50}$$

where E_1 and E_2 are both convex, E_2 is differentiable, and $\Omega \subset X$ is a convex set. Let

$$g(\boldsymbol{\xi}, \boldsymbol{\xi}_t) = E_1(\boldsymbol{\xi}) - E_2(\boldsymbol{\xi}_t) - (\nabla_{\boldsymbol{\xi}} E_2(\boldsymbol{\xi}_t))^T (\boldsymbol{\xi} - \boldsymbol{\xi}_t) \tag{51}$$

Then the MM (Majorization-Minimization) algorithm [3]

$$\boldsymbol{\xi}_{t+1} \in \mathcal{A}(\boldsymbol{\xi}_t) = \arg\min_{\boldsymbol{\xi} \in \Omega} g(\boldsymbol{\xi}, \boldsymbol{\xi}_t)$$
$$= \arg\min_{\boldsymbol{\xi} \in \Omega} \left(E_1(\boldsymbol{\xi}) - (\nabla_{\boldsymbol{\xi}} E_2(\boldsymbol{\xi}_t))^T \boldsymbol{\xi} \right) \tag{52}$$
$$t = 0, 1, \cdots$$

guarantees $\mathcal{A}$ to be monotonic with respect to E.
In addition, if E_1 is strictly convex, then $\mathcal{A}$ is strictly monotonic with respect to E.

Proof. Since E_2 is convex and differentiable, then the first order characterization of convexity holds:

$$E_2(\boldsymbol{\xi}) \geqslant E_2(\boldsymbol{\xi}_t) + (\nabla_{\boldsymbol{\xi}} E_2(\boldsymbol{\xi}_t))^T (\boldsymbol{\xi} - \boldsymbol{\xi}_t) \tag{53}$$

Therefore:

$$E(\boldsymbol{\xi}) = E_1(\boldsymbol{\xi}) - E_2(\boldsymbol{\xi})$$
$$\leqslant E_1(\boldsymbol{\xi}) - E_2(\boldsymbol{\xi}_t) - (\nabla_{\boldsymbol{\xi}} E_2(\boldsymbol{\xi}_t))^T (\boldsymbol{\xi} - \boldsymbol{\xi}_t) \tag{54}$$
$$= g(\boldsymbol{\xi}, \boldsymbol{\xi}_t)$$

According to 52

$$\forall \boldsymbol{\xi}_{t+1} \in \mathcal{A}(\boldsymbol{\xi}_t), g(\boldsymbol{\xi}_{t+1}, \boldsymbol{\xi}_t) \leqslant g(\boldsymbol{\xi}_t, \boldsymbol{\xi}_t) \tag{55}$$

Using 51, 54, 55

$$\forall \boldsymbol{\xi}_{t+1} \in \mathcal{A}(\boldsymbol{\xi}_t), E(\boldsymbol{\xi}_{t+1}) \leqslant g(\boldsymbol{\xi}_{t+1}, \boldsymbol{\xi}_t) \leqslant g(\boldsymbol{\xi}_t, \boldsymbol{\xi}_t) = E(\boldsymbol{\xi}_t) \tag{56}$$

Thus

13

$$\forall \boldsymbol{\xi}_{t+1} \in \mathcal{A}\left(\boldsymbol{\xi}_t\right), \mathrm{E}\left(\boldsymbol{\xi}_{t+1}\right) \leqslant \mathrm{E}\left(\boldsymbol{\xi}_t\right) \tag{57}$$

Therefore $\mathcal{A}$ is monotonic with respect to E.

If E_1 is strictly convex with respect to $\boldsymbol{\xi}$, then $g\left(\boldsymbol{\xi}, \boldsymbol{\xi}_t\right)$ is also strictly convex with respect to $\boldsymbol{\xi}$ according to its definition 51. Since Ω is convex, then there exists only one minimum in 52, which is the global minimum.

Hence

$$\forall \boldsymbol{\xi}_{t+1} \in \mathcal{A}\left(\boldsymbol{\xi}_t\right), \boldsymbol{\xi}_{t+1} \neq \boldsymbol{\xi}_t \implies g\left(\boldsymbol{\xi}_{t+1}, \boldsymbol{\xi}_t\right) < g\left(\boldsymbol{\xi}_t, \boldsymbol{\xi}_t\right) \tag{58}$$

Using 51, 54, 58

$$\begin{aligned} &\forall \boldsymbol{\xi}_{t+1} \in \mathcal{A}\left(\boldsymbol{\xi}_t\right), \boldsymbol{\xi}_{t+1} \neq \boldsymbol{\xi}_t \implies \\ &\mathrm{E}\left(\boldsymbol{\xi}_{t+1}\right) \leqslant g\left(\boldsymbol{\xi}_{t+1}, \boldsymbol{\xi}_t\right) < g\left(\boldsymbol{\xi}_t, \boldsymbol{\xi}_t\right) = \mathrm{E}\left(\boldsymbol{\xi}_t\right) \end{aligned} \tag{59}$$

That is

$$\forall \boldsymbol{\xi}_{t+1} \in \mathcal{A}\left(\boldsymbol{\xi}_t\right), \boldsymbol{\xi}_{t+1} \neq \boldsymbol{\xi}_t \implies \mathrm{E}\left(\boldsymbol{\xi}_{t+1}\right) < \mathrm{E}\left(\boldsymbol{\xi}_t\right) \tag{60}$$

If $\mathcal{A}\left(\boldsymbol{\xi}_t\right) \neq \{\boldsymbol{\xi}_t\}$, suppose $\boldsymbol{\xi}_t \in \mathcal{A}\left(\boldsymbol{\xi}_t\right)$, then $\exists \boldsymbol{\xi}'_{t+1} \in \mathcal{A}\left(\boldsymbol{\xi}_t\right), \boldsymbol{\xi}'_{t+1} \neq \boldsymbol{\xi}_t$. Using 58

$$g\left(\boldsymbol{\xi}'_{t+1}, \boldsymbol{\xi}_t\right) < g\left(\boldsymbol{\xi}_t, \boldsymbol{\xi}_t\right) \tag{61}$$

Hence

$$\boldsymbol{\xi}_{t+1} \notin \mathcal{A}\left(\boldsymbol{\xi}_t\right) = \arg\min_{\boldsymbol{\xi} \in \Omega} g\left(\boldsymbol{\xi}, \boldsymbol{\xi}_t\right) \tag{62}$$

which is a contradiction. Thus if $\mathcal{A}\left(\boldsymbol{\xi}_t\right) \neq \{\boldsymbol{\xi}_t\}$, then $\boldsymbol{\xi}_{t+1} \notin \mathcal{A}\left(\boldsymbol{\xi}_t\right)$. That is $\forall \boldsymbol{\xi}_{t+1} \in \mathcal{A}\left(\boldsymbol{\xi}_t\right), \boldsymbol{\xi}_{t+1} \neq \boldsymbol{\xi}_t$. Using 60, $\mathrm{E}\left(\boldsymbol{\xi}_{t+1}\right) < \mathrm{E}\left(\boldsymbol{\xi}_t\right)$. It follows that

$$\mathcal{A}\left(\boldsymbol{\xi}_t\right) \neq \{\boldsymbol{\xi}_t\} \implies \forall \boldsymbol{\xi}_{t+1} \in \mathcal{A}\left(\boldsymbol{\xi}_t\right), \mathrm{E}\left(\boldsymbol{\xi}_{t+1}\right) < \mathrm{E}\left(\boldsymbol{\xi}_t\right) \tag{63}$$

Therefore point-set-map $\mathcal{A}$ is strictly monotonic with respect to E.

$$\square$$

19 Lemma 11 Hopfield MM Algorithm Update Rule

$$\arg\min_{\boldsymbol{\xi}} g\left(\boldsymbol{\xi}, \boldsymbol{\xi}_t\right) = \arg\min_{\boldsymbol{\xi} \in S} g\left(\boldsymbol{\xi}, \boldsymbol{\xi}_t\right) = \left\{ \boldsymbol{X} \operatorname{softmax}\left(\beta \boldsymbol{X}^T \boldsymbol{\xi}_t\right) \right\} \tag{64}$$

where g is defined by 51, $\mathrm{E}_1, \mathrm{E}_2$ are defined by 31, S is a convex and compact set defined by 32.

Proof. E_1 is strictly convex with respect to $\boldsymbol{\xi}$, then $g\left(\boldsymbol{\xi}, \boldsymbol{\xi}_t\right)$ is also strictly convex with respect to $\boldsymbol{\xi}$ according to its definition 51. Then there exists only one minimum in 52, which is the global minimum.
Let

$$\nabla_{\boldsymbol{\xi}} g\left(\boldsymbol{\xi}, \boldsymbol{\xi}_t\right) = \boldsymbol{0} \tag{65}$$

Then

$$\nabla_{\boldsymbol{\xi}} \mathrm{E}_1\left(\boldsymbol{\xi}\right) = \nabla_{\boldsymbol{\xi}} \mathrm{E}_2\left(\boldsymbol{\xi}_t\right) \tag{66}$$

Using 4, 31

$$\boldsymbol{\xi} = \boldsymbol{X} \operatorname{softmax}\left(\beta \boldsymbol{X}^T \boldsymbol{\xi}_t\right) \tag{67}$$

Using 33

$$\boldsymbol{\xi} \in S \tag{68}$$

Therefore

$$\arg\min_{\boldsymbol{\xi}} g\left(\boldsymbol{\xi}, \boldsymbol{\xi}_t\right) = \arg\min_{\boldsymbol{\xi} \in S} g\left(\boldsymbol{\xi}, \boldsymbol{\xi}_t\right) = \left\{ \boldsymbol{X} \operatorname{softmax}\left(\beta \boldsymbol{X}^T \boldsymbol{\xi}_t\right) \right\} \tag{69}$$

$\square$

20 Lemma 12 Closedness Sufficient Condition

Given a continuous function $h\left(\mathbf{x}, \mathbf{y}\right)$ on $X \times Y$, where X and Y are closed sets, define the point-to-set map $\mathcal{A} : X \to \mathscr{P}(Y)$ by

$$\mathcal{A}\left(\mathbf{x}\right) = \arg\min_{\mathbf{y} \in Y} h\left(\mathbf{x}, \mathbf{y}\right) \tag{70}$$

If $\mathcal{A}$ is nonempty at each $\mathbf{x} \in X$, then $\mathcal{A}$ is closed on X [4].

Proof. According to closedness definition, suppose

$$
\begin{cases}
\mathbf{x}_t \to \mathbf{x}_0 \text{ as } t \to \infty \\
\mathbf{y}_t \to \mathbf{y}_0 \text{ as } t \to \infty \\
\mathbf{x}_t \in X \\
\mathbf{y}_t \in \mathcal{A}(\mathbf{x}_t) = \arg\min_{\mathbf{y} \in Y} h(\mathbf{x}_t, \mathbf{y})
\end{cases}
\tag{71}
$$

Since X and Y are closed sets, $(\mathbf{x}_0, \mathbf{y}_0) \in X \times Y$.
By continuity of h

$$
h(\mathbf{x}_t, \mathbf{y}_t) \to h(\mathbf{x}_0, \mathbf{y}_0) \text{ as } t \to \infty
$$
$$
\forall \mathbf{y} \in Y, h(\mathbf{x}_t, \mathbf{y}) \to h(\mathbf{x}_0, \mathbf{y}) \text{ as } t \to \infty
\tag{72}
$$

Using the definition of limit, $\forall \mathbf{y} \in Y, \forall \varepsilon_2 > 0$, there is a t_{ε_2} such that $\forall t > t_{\varepsilon_2}$,

$$
h(\mathbf{x}_t, \mathbf{y}_t) > h(\mathbf{x}_0, \mathbf{y}_0) - \varepsilon_2
$$
$$
h(\mathbf{x}_t, \mathbf{y}) < h(\mathbf{x}_0, \mathbf{y}) + \varepsilon_2
\tag{73}
$$

and

$$
\mathbf{y}_t \in \mathcal{A}(\mathbf{x}_t) = \arg\min_{\mathbf{y} \in Y} h(\mathbf{x}_t, \mathbf{y}) \implies h(\mathbf{x}_t, \mathbf{y}_t) \leqslant h(\mathbf{x}_t, \mathbf{y})
\tag{74}
$$

Using 73, 74

$$
h(\mathbf{x}_0, \mathbf{y}_0) < h(\mathbf{x}_t, \mathbf{y}_t) + \varepsilon_2 \leqslant h(\mathbf{x}_t, \mathbf{y}) + \varepsilon_2 < h(\mathbf{x}_0, \mathbf{y}) + 2\varepsilon_2
\tag{75}
$$

Hence

$$
h(\mathbf{x}_0, \mathbf{y}_0) \leqslant h(\mathbf{x}_0, \mathbf{y})
\tag{76}
$$

It follows that

$$
\mathbf{y}_0 \in \mathcal{A}(\mathbf{x}_0) = \arg\min_{\mathbf{y} \in Y} h(\mathbf{x}_0, \mathbf{y})
\tag{77}
$$

Therefore $\mathcal{A}$ is closed on X.

$\square$

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21 Lemma 13 Convergence Theorem 2

Let a point-to-set map $\mathcal{A} : X \to \mathscr{P}(X)$ be a point-to-set map (an algorithm) that given a point $\boldsymbol{\xi}_0 \in X$ generates a sequence $\{\boldsymbol{\xi}_t\}_{t=0}^{\infty}$ through the iteration $\boldsymbol{\xi}_{t+1} \in \mathcal{A}(\boldsymbol{\xi}_t)$. Also let a solution set $\Gamma \subset X$ be given. Suppose

1. All points $\boldsymbol{\xi}_{t+1}, t \geqslant 0$ are in a compact set $\Omega \subset X$.

2. There is a continuous function $E : X \to \mathbb{R}$ such that:

   (a) $\boldsymbol{\xi}_t \notin \Gamma \implies E\left(\boldsymbol{\xi}_{t+1}\right) < E\left(\boldsymbol{\xi}_t\right), \forall \boldsymbol{\xi}_{t+1} \in \mathcal{A}(\boldsymbol{\xi}_t)$

   (b) $\boldsymbol{\xi}_t \in \Gamma \implies E\left(\boldsymbol{\xi}_{t+1}\right) \leqslant E\left(\boldsymbol{\xi}_t\right), \forall \boldsymbol{\xi}_{t+1} \in \mathcal{A}(\boldsymbol{\xi}_t)$

3. $\mathcal{A}$ is closed on X.

Then all limit points of the sequence $\{\boldsymbol{\xi}_t\}_{t=0}^{\infty}$ are in Γ [5].

Proof. Given any convergent subsequence of $\{\boldsymbol{\xi}_t\}_{t=0}^{\infty}$

$$\boldsymbol{\xi}_{t_k} \to \boldsymbol{\xi}^*, \text{ as } k \to \infty \tag{78}$$

where $\boldsymbol{\xi}^* \in \Omega$ is a limit point, consider subsequence:

$$\left\{\boldsymbol{\xi}_{t_k+1}\right\}_{k=0}^{\infty} \tag{79}$$

Since all points $\boldsymbol{\xi}_{t+1}, t \geqslant 0$ are in the compact set Ω, then $\left\{\boldsymbol{\xi}_{t_k+1}\right\}_{k=0}^{\infty}$ must have a convergent subsequence

$$\boldsymbol{\xi}_{t_{k_l}+1} \to \boldsymbol{\xi}^{**}, \text{ as } l \to \infty \tag{80}$$

where $\boldsymbol{\xi}^{**} \in \Omega$ is a limit point.
Using 78

$$\boldsymbol{\xi}_{t_{k_l}} \to \boldsymbol{\xi}^*, \text{ as } l \to \infty \tag{81}$$

It is known that

$$\boldsymbol{\xi}_{t_{k_l}+1} \in \mathcal{A}\left(\boldsymbol{\xi}_{t_{k_l}}\right) \tag{82}$$

Since $\mathcal{A}$ is closed on X, then using 80, 81 and 82

$$\boldsymbol{\xi}^{**} \in \mathcal{A}(\boldsymbol{\xi}^*) \tag{83}$$

Since

1. All points $\boldsymbol{\xi}_{t+1}, t \geqslant 0$ are in the compact set Ω.

2. $\mathcal{A}$ is monotonic with respect to a continuous function E.

According to Lemma 9,

$$\lim_{t \to \infty} \mathrm{E}\left(\boldsymbol{\xi}_t\right) = \mathrm{E}\left(\boldsymbol{\xi}^{**}\right) = \mathrm{E}\left(\boldsymbol{\xi}^*\right) \tag{84}$$

If $\boldsymbol{\xi}^* \notin \Gamma$, then $\mathrm{E}\left(\boldsymbol{\xi}^{**}\right) < \mathrm{E}\left(\boldsymbol{\xi}^*\right)$. A contradiction. Therefore $\boldsymbol{\xi}^* \in \Gamma$.

$\square$

22 Lemma 14 Fixed Points Sufficient Condition

Let a point-to-set map $\mathcal{A} : X \to \mathscr{P}(X)$ be a point-to-set map (an algorithm) that given a point $\boldsymbol{\xi}_0 \in X$ generates a sequence $\{\boldsymbol{\xi}_t\}_{t=0}^{\infty}$ through the iteration $\boldsymbol{\xi}_{t+1} \in \mathcal{A}\left(\boldsymbol{\xi}_t\right)$. Suppose

1. All points $\boldsymbol{\xi}_{t+1}, t \geqslant 0$ are in a compact set $\Omega \subset X$.

2. $\mathcal{A}$ is strictly monotonic with respect to a continuous function E.

3. $\mathcal{A}$ is closed on X.

Then all limit points of the sequence $\{\boldsymbol{\xi}_t\}_{t=0}^{\infty}$ are fixed points of $\mathcal{A}$.

Proof. Let

$$\Gamma = \{\boldsymbol{\xi} \in X | \mathcal{A}\left(\boldsymbol{\xi}\right) = \{\boldsymbol{\xi}\}\} \tag{85}$$

Using Lemma 13, all limit points of the sequence $\{\boldsymbol{\xi}_t\}_{t=0}^{\infty}$ are in Γ.
Therefore all limit points are fixed points of $\mathcal{A}$.

$\square$

23 Lemma 15 Stationary Points Sufficient Condition

Suppose $\mathrm{E}_1\left(\boldsymbol{\xi}\right), \mathrm{E}_2\left(\boldsymbol{\xi}\right)$ are differentiable.
And

$$\Omega = \{\boldsymbol{\xi} : c_i\left(\boldsymbol{\xi}\right) \leqslant 0, i \in [m], d_j\left(\boldsymbol{\xi}\right) = 0, j \in [p]\} \tag{86}$$

$\boldsymbol{\xi}^*$ is a generalized fixed point of 52, then $\boldsymbol{\xi}^*$ is a stationary point of the the program in 50.

Proof. We have $\boldsymbol{\xi}^* \in \mathcal{A}(\boldsymbol{\xi}^*)$ and $\boldsymbol{\xi}^* \in \Omega$. Then there exist Lagrange multipliers $\{\eta_i^*\}_{i=1}^m \subset \mathbb{R}^+$ and $\{\mu_j^*\}_{j=1}^p \subset \mathbb{R}$ such that the following KKT conditions hold:

$$
\begin{cases}
\nabla_{\boldsymbol{\xi}} \mathrm{E}_1(\boldsymbol{\xi}^*) - \nabla_{\boldsymbol{\xi}} \mathrm{E}_2(\boldsymbol{\xi}^*) + \sum_{i=1}^m \eta_i^* \nabla_{\boldsymbol{\xi}} c_i(\boldsymbol{\xi}^*) + \sum_{j=1}^p \mu_j^* \nabla_{\boldsymbol{\xi}} d_j(\boldsymbol{\xi}^*) = \mathbf{0} \\
c_i(\boldsymbol{\xi}^*) \leqslant 0, \eta_i^* \geqslant 0, c_i(\boldsymbol{\xi}^*) \eta_i^* = 0, \forall i \in [m] \\
d_j(\boldsymbol{\xi}^*) = 0, \mu_j^* \in \mathbb{R}, \forall j \in [p]
\end{cases}
\tag{87}
$$

is exactly KKT conditions of 50 which are satisfied by $\left(\boldsymbol{\xi}^*, \{\eta_i^*\}_{i=1}^m, \{\mu_j^*\}_{j=1}^p\right)$ and therefore, is a stationary point of 50.

$\square$

24 Lemma 16 Convergence of Adjacent Point Differences

Let a point-to-set map $\mathcal{A} : X \to \mathscr{P}(X)$ be a point-to-set map (an algorithm) that given a point $\boldsymbol{\xi}_0 \in X$ generates a sequence $\{\boldsymbol{\xi}_t\}_{t=0}^\infty$ through the iteration $\boldsymbol{\xi}_{t+1} \in \mathcal{A}(\boldsymbol{\xi}_t)$. Suppose [6]

1. All points $\boldsymbol{\xi}_{t+1}, t \geqslant 0$ are in a compact set $\Omega \subset X$.

2. $\mathcal{A}$ is strictly monotonic with respect to a continuous function E.

3. $\mathcal{A}$ is closed on X.

Then

$$
\left\|\boldsymbol{\xi}_{t+1} - \boldsymbol{\xi}_t\right\|_2 \to 0, \text{ as } t \to \infty
\tag{88}
$$

Proof. Suppose

$$
\left\|\boldsymbol{\xi}_{t+1} - \boldsymbol{\xi}_t\right\|_2 \nrightarrow 0, \text{ as } t \to \infty
\tag{89}
$$

Then there exists a $\varepsilon_3 > 0$ and a subsequence

$$
\left\{\boldsymbol{\xi}_{t_k}\right\}_{k=0}^\infty
\tag{90}
$$

such that

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$$\left\| \boldsymbol{\xi}_{t_k+1} - \boldsymbol{\xi}_{t_k} \right\|_2 \geqslant \varepsilon_3, \forall k \geqslant 0 \tag{91}$$

Since all points $\boldsymbol{\xi}_{t+1}, t \geqslant 0$ are in a compact set Ω, then $\left\{ \boldsymbol{\xi}_{t_k} \right\}_{k=0}^{\infty}$ must have a convergent subsequence

$$\boldsymbol{\xi}_{t_{k_l}} \to \boldsymbol{\xi}^*, \text{ as } l \to \infty \tag{92}$$

where $\boldsymbol{\xi}^* \in \Omega$ is a limit point.
Consider subsequence:

$$\left\{ \boldsymbol{\xi}_{t_{k_l}+1} \right\}_{k=0}^{\infty} \tag{93}$$

It must have a convergent subsequence

$$\boldsymbol{\xi}_{t_{k_{l_m}}+1} \to \boldsymbol{\xi}^{**}, \text{ as } m \to \infty \tag{94}$$

where $\boldsymbol{\xi}^{**} \in \Omega$ is a limit point.
Using 92

$$\boldsymbol{\xi}_{t_{k_{l_m}}} \to \boldsymbol{\xi}^*, \text{ as } m \to \infty \tag{95}$$

It is known that

$$\boldsymbol{\xi}_{t_{k_{l_m}}+1} \in \mathcal{A}\left(\boldsymbol{\xi}_{t_{k_{l_m}}} \right) \tag{96}$$

Since $\mathcal{A}$ is closed on X, using 94, 95 and 96

$$\boldsymbol{\xi}^{**} \in \mathcal{A}\left(\boldsymbol{\xi}^* \right) \tag{97}$$

Since

1. All points $\boldsymbol{\xi}_{t+1}, t \geqslant 0$ are in the compact set Ω.

2. $\mathcal{A}$ is monotonic with respect to a continuous function E.

According to Lemma 9,

$$\lim_{t \to \infty} \mathrm{E}\left(\boldsymbol{\xi}_t \right) = \mathrm{E}\left(\boldsymbol{\xi}^{**} \right) = \mathrm{E}\left(\boldsymbol{\xi}^* \right) \tag{98}$$

Since $\mathcal{A}$ is strictly monotonic with respect to E, using 97, 98

20

$$\mathcal{A}\left(\boldsymbol{\xi}^*\right) = \{\boldsymbol{\xi}^*\} \tag{99}$$

Hence

$$\boldsymbol{\xi}^* = \boldsymbol{\xi}^{**} \tag{100}$$

using 94, 95 and 100

$$\left\|\boldsymbol{\xi}_{t_{k_{l_m}}+1} - \boldsymbol{\xi}_{t_{k_{l_m}}}\right\|_2 \to 0, \text{ as } m \to \infty \tag{101}$$

which is in contradiction with 91.
Therefore

$$\left\|\boldsymbol{\xi}_{t+1} - \boldsymbol{\xi}_t\right\|_2 \to 0, \text{ as } t \to \infty \tag{102}$$

$\square$

25 Lemma 17 Compactness of Limit Points

Let a point-to-set map $\mathcal{A} : X \to \mathscr{P}(X)$ be a point-to-set map (an algorithm) that given a point $\boldsymbol{\xi}_0 \in X$ generates a sequence $\{\boldsymbol{\xi}_t\}_{t=0}^{\infty}$ through the iteration $\boldsymbol{\xi}_{t+1} \in \mathcal{A}\left(\boldsymbol{\xi}_t\right)$.
Suppose all points $\boldsymbol{\xi}_{t+1}, t \geqslant 0$ are in a compact set $\Omega \subset X$.
Let S_{lim} denotes the set of limit points of $\{\boldsymbol{\xi}_t\}_{t=0}^{\infty}$.
Then S_{lim} is compact.

Proof. Since all points $\boldsymbol{\xi}_{t+1}, t \geqslant 0$ are in a compact set Ω, then S_{lim} is not empty and

$$\forall \boldsymbol{\xi}_{lim} \in S_{lim}, \boldsymbol{\xi}_{lim} \in \Omega \tag{103}$$

Thus S_{lim} is bounded.
For any convergent subsequence of S_{lim}

$$\left\{\boldsymbol{\xi}_{lim_k}\right\}_{k=0}^{\infty} \tag{104}$$

suppose

$$\boldsymbol{\xi}_{lim_k} \to \boldsymbol{\xi}_{lim}^* \text{ as } k \to \infty \tag{105}$$

Since $\forall k \geqslant 0, \boldsymbol{\xi}_{lim_k}$ is a limit point of $\{\boldsymbol{\xi}_t\}_{t=0}^{\infty}$, using the definition of limit,

$$\forall k \geqslant 0, \exists \boldsymbol{\xi}_{t_k} \in \{\boldsymbol{\xi}_t\}_{t=0}^{\infty}, \left\|\boldsymbol{\xi}_{t_k} - \boldsymbol{\xi}_{lim_k}\right\|_2 < \frac{1}{k+1} \tag{106}$$

Then

$$\left\|\boldsymbol{\xi}_{t_k} - \boldsymbol{\xi}_{lim_k}\right\|_2 \to 0, k \to \infty \tag{107}$$

That is

$$\boldsymbol{\xi}_{t_k} - \boldsymbol{\xi}_{lim_k} \to \mathbf{0}, k \to \infty \tag{108}$$

Using 105, 108

$$\boldsymbol{\xi}_{t_k} = \left(\boldsymbol{\xi}_{t_k} - \boldsymbol{\xi}_{lim_k}\right) + \boldsymbol{\xi}_{lim_k} \to \boldsymbol{\xi}_{lim}^*, k \to \infty \tag{109}$$

Since S_{lim} denotes the set of limit points of $\{\boldsymbol{\xi}_t\}_{t=0}^{\infty}$, it follows that

$$\boldsymbol{\xi}^* \in S_{lim} \tag{110}$$

Therefore S_{lim} is closed. It is also bounded, so it is a compact set.

$\square$

26   Lemma 18 Connectedness of Limit Points

Let a point-to-set map $\mathcal{A} : X \to \mathscr{P}(X)$ be a point-to-set map (an algorithm) that given a point $\boldsymbol{\xi}_0 \in X$ generates a sequence $\{\boldsymbol{\xi}_t\}_{t=0}^{\infty}$ through the iteration $\boldsymbol{\xi}_{t+1} \in \mathcal{A}(\boldsymbol{\xi}_t)$.
Suppose

1. All points $\boldsymbol{\xi}_{t+1}, t \geqslant 0$ are in a compact set $\Omega \subset X$.

2. $\left\|\boldsymbol{\xi}_{t+1} - \boldsymbol{\xi}_t\right\|_2 \to 0$, as $t \to \infty$

3. $\{\boldsymbol{\xi}_t\}_{t=0}^{\infty}$ does not converge.

Let S_{lim} denotes the set of limit points of $\{\boldsymbol{\xi}_t\}_{t=0}^{\infty}$.
Then S_{lim} is connected.

Proof. Since $\{\boldsymbol{\xi}_t\}_{t=0}^{\infty}$ does not converge, then S_{lim} must contain at least two points. Suppose S_{lim} is not connected. Then it can be decomposed into the union of two nonempty closed sets of points without common points.

1. **Define Two Subsets**

    Let

$$S_{lim} = S_{lim}^1 \cup S_{lim}^2 \tag{111}$$

    where $S_{lim}^1 \cup S_{lim}^2$ are both nonempty, closed and:

$$S_{lim}^1 \cap S_{lim}^2 = \varnothing \tag{112}$$

2. **Define Distance to Set**

    Let

$$d = \inf \left\{ \left\| \boldsymbol{\xi}_{lim}^1 - \boldsymbol{\xi}_{lim}^2 \right\|_2 \big| \boldsymbol{\xi}_{lim}^1 \in S_{lim}^1, \boldsymbol{\xi}_{lim}^2 \in S_{lim}^2 \right\} \tag{113}$$

    Suppose $d = 0$, then there is a sequence $\left\{ \boldsymbol{\xi}_{lim_k}^1 \right\}_{k=1}^{\infty}$ that belongs to S_{lim}^1, and a sequence $\left\{ \boldsymbol{\xi}_{lim_k}^2 \right\}_{k=1}^{\infty}$ that belongs to S_{lim}^2, and

$$\boldsymbol{\xi}_{lim_k}^1 - \boldsymbol{\xi}_{lim_k}^2 \to \mathbf{0}, k \to \infty \tag{114}$$

    Since S_{lim}^1 and S_{lim}^2 are closed, then each of the sequences $\left\{ \boldsymbol{\xi}_{lim_k}^1 \right\}_{k=1}^{\infty}$ and $\left\{ \boldsymbol{\xi}_{lim_k}^2 \right\}_{k=1}^{\infty}$ has a convergent subsequence. Using 114, two subsequences converge to the same limit point. Then this limit point belongs to both closed set S_{lim}^1 and S_{lim}^2, which is a contradiction with 112. Thus $d > 0$.

    Since S_{lim}^1 and S_{lim}^2 are nonempty, then there exists two limit points $\boldsymbol{\xi}_{lim}^{*1} \in S_{lim}^1, \boldsymbol{\xi}_{lim}^{*2} \in S_{lim}^2$.

    Suppose

$$\begin{cases} \boldsymbol{\xi}_{t_k}^1 \to \boldsymbol{\xi}_{lim}^{*1}, \text{ as } k \to \infty \\ \boldsymbol{\xi}_{t_k}^2 \to \boldsymbol{\xi}_{lim}^{*2}, \text{ as } k \to \infty \end{cases} \tag{115}$$

    where $Q_1 = \left\{ \boldsymbol{\xi}_{t_k}^1 \right\}_{k=0}^{\infty}$ and $Q_2 = \left\{ \boldsymbol{\xi}_{t_k}^2 \right\}_{k=0}^{\infty}$ are subsequences of $\left\{ \boldsymbol{\xi}_t \right\}_{t=0}^{\infty}$.

    Since

$$\left\| \boldsymbol{\xi}_{t+1} - \boldsymbol{\xi}_t \right\|_2 \to 0, \text{ as } t \to \infty \tag{116}$$

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then

$$\exists t_d, \forall t > t_d, \left\|\boldsymbol{\xi}_{t+1} - \boldsymbol{\xi}_t\right\|_2 < \frac{d}{3} \tag{117}$$

Let

$$\begin{cases} d_1\left(\boldsymbol{\xi}\right) = \inf\left\{\left\|\boldsymbol{\xi} - \boldsymbol{\xi}_{lim}^1\right\|_2 \middle| \boldsymbol{\xi} \in \{\boldsymbol{\xi}_t\}_{t=0}^{\infty}, \boldsymbol{\xi}_{lim}^1 \in S_{lim}^1\right\} \\ d_2\left(\boldsymbol{\xi}\right) = \inf\left\{\left\|\boldsymbol{\xi} - \boldsymbol{\xi}_{lim}^2\right\|_2 \middle| \boldsymbol{\xi} \in \{\boldsymbol{\xi}_t\}_{t=0}^{\infty}, \boldsymbol{\xi}_{lim}^2 \in S_{lim}^2\right\} \end{cases} \tag{118}$$

Using 113, 118

$$\forall \boldsymbol{\xi} \in \{\boldsymbol{\xi}_t\}_{t=0}^{\infty}, d_1\left(\boldsymbol{\xi}\right) + d_2\left(\boldsymbol{\xi}\right) \geqslant d \tag{119}$$

3. **Create Subsequence**

   Now let's create a subsequence of $\{\boldsymbol{\xi}_t\}_{t=0}^{\infty}$ whose limit points are outside both S_{lim}^1 and S_{lim}^2:

   For any arbitrarily large number K, Using 115, 118

$$\begin{cases} \exists m > \max\{K, t_d\}, d_1\left(\boldsymbol{\xi}_m\right) < \frac{d}{3} \\ \exists n > m, d_2\left(\boldsymbol{\xi}_n\right) < \frac{d}{3} \end{cases} \tag{120}$$

   Define index set

$$I = \{i | m \leqslant i \leqslant n\} \tag{121}$$

   Using 119, 120

$$\begin{cases} d_2\left(\boldsymbol{\xi}_m\right) \geqslant \frac{2}{3}d \\ d_1\left(\boldsymbol{\xi}_n\right) \geqslant \frac{2}{3}d \end{cases} \tag{122}$$

   Using 120, 121, 122

$$\exists t_{k_1} = \min\left\{i \in I \middle| d_1\left(\boldsymbol{\xi}_i\right) \geqslant \frac{d}{3}\right\} \tag{123}$$

24

Obviously:

$$\begin{cases} t_{k_1} > m \\ d_1 \left(\boldsymbol{\xi}_{t_{k_1}} \right) \geqslant \dfrac{d}{3} \\ d_1 \left(\boldsymbol{\xi}_{t_{k_1}-1} \right) < \dfrac{d}{3} \end{cases} \tag{124}$$

Using 119, 124

$$d_2 \left(\boldsymbol{\xi}_{t_{k_1}-1} \right) \geqslant \frac{2}{3}d \tag{125}$$

Using 117, 125

$$d_2 \left(\boldsymbol{\xi}_{t_{k_1}} \right) \geqslant \frac{d}{3} \tag{126}$$

Using 124, 126

$$\left. \begin{matrix} \boldsymbol{\xi}_{t_{k_1}} \notin S_{lim}^1 \\ \boldsymbol{\xi}_{t_{k_1}} \notin S_{lim}^2 \end{matrix} \right\} \implies \boldsymbol{\xi}_{t_{k_1}} \notin S_{lim}^1 \cup S_{lim}^2 \tag{127}$$

Follow this way, there exists an infinite sequence of $k_1, k_2, \cdots$ for which 127 holds. Therefore all limit points of this sequence are outside both S_{lim}^1 and S_{lim}^2, which is in contradiction with 111. Therefore S_{lim} is connected.

$\square$

27 Theorem 2 Global Convergence: Stationary Points

For the iteration 23 we have

$$\mathrm{E} \left(\boldsymbol{\xi}_t \right) \to \mathrm{E} \left(\boldsymbol{\xi}^* \right) = \mathrm{E}^*, \text{ as } t \to \infty \tag{128}$$

for some stationary point $\boldsymbol{\xi}^*$. Furthermore

$$\left\| \boldsymbol{\xi}_{t+1} - \boldsymbol{\xi}_t \right\|_2 \to 0, \text{ as } t \to \infty \tag{129}$$

And either $\{\boldsymbol{\xi}_t\}_{t=0}^{\infty}$ converges, or, in the other case, the set of limit points of $\{\boldsymbol{\xi}_t\}_{t=0}^{\infty}$ is a connected and compact subset of $\mathcal{L}(\mathrm{E}^*)$, where $\mathcal{L}(a) = \{\boldsymbol{\xi} \in \mathcal{L} | \mathrm{E}(\boldsymbol{\xi}) = a\}$ and $\mathcal{L}$ is the set of stationary points of the iteration 23. If $\mathcal{L}(\mathrm{E}^*)$ is finite, then any sequence $\{\boldsymbol{\xi}_t\}_{t=0}^{\infty}$ generated by the iteration 23 converges to some $\boldsymbol{\xi}^* \in \mathcal{L}(\mathrm{E}^*)$.

Proof. Define minimization problem:

$$\min_{\boldsymbol{\xi}} \mathrm{E}(\boldsymbol{\xi}) = \min_{\boldsymbol{\xi}} \left(\mathrm{E}_1(\boldsymbol{\xi}) - \mathrm{E}_2(\boldsymbol{\xi}) \right) \tag{130}$$

where $\mathrm{E}_1, \mathrm{E}_2$ are defined by 31.

1. **Applying MM Algorithm**

   According to 51, 52, let

   $$g(\boldsymbol{\xi}, \boldsymbol{\xi}_t) = \mathrm{E}_1(\boldsymbol{\xi}) - \mathrm{E}_2(\boldsymbol{\xi}_t) - \left(\nabla_{\boldsymbol{\xi}} \mathrm{E}_2(\boldsymbol{\xi}_t) \right)^T (\boldsymbol{\xi} - \boldsymbol{\xi}_t) \tag{131}$$

   $$\boldsymbol{\xi}_{t+1} \in \mathcal{A}(\boldsymbol{\xi}_t) = \arg\min_{\boldsymbol{\xi}} g(\boldsymbol{\xi}, \boldsymbol{\xi}_t)$$

   $$= \arg\min_{\boldsymbol{\xi}} g\left(\mathrm{E}_1(\boldsymbol{\xi}) - \left(\nabla_{\boldsymbol{\xi}} \mathrm{E}_2(\boldsymbol{\xi}_t) \right)^T \boldsymbol{\xi} \right) \tag{132}$$

   $$t = 0, 1, \cdots$$

   According to 64, the resulting update rule is:

   $$\boldsymbol{\xi}_{t+1} = \boldsymbol{X} \operatorname{softmax}\left(\beta \boldsymbol{X}^T \boldsymbol{\xi}_t \right) \tag{133}$$

   which is equivalent to 23.

2. $\mathcal{A}$ **is Strictly Monotonic**

   $\mathrm{E}_1(\boldsymbol{\xi}) = \frac{1}{2} \boldsymbol{\xi}^T \boldsymbol{\xi} + C_1$ is strictly convex.

   $\mathrm{E}_2(\boldsymbol{\xi})$ is convex and differentiable using 4, 31.

   Then using Lemma 10, $\mathcal{A}$ is strictly monotonic with respect to E.

3. $\mathcal{A}$ **is Uniformly Compact**

Using 64

All points $\boldsymbol{\xi}_{t+1}, t \geqslant 0$ are in the compact set S using 64, therefore $\mathcal{A}$ is uniformly compact on $\mathbb{R}^d$.

4. $\mathcal{A}$ **is Closed**

Let

$$h\left(\mathbf{x}, \mathbf{y}\right) = g\left(\mathbf{y}, \mathbf{x}\right) \tag{134}$$

It follows that

$$\mathcal{A}\left(\boldsymbol{\xi}\right) = \arg\min_{\boldsymbol{\xi}} g\left(\boldsymbol{\xi}, \boldsymbol{\xi}_t\right) = \arg\min_{\boldsymbol{\xi}} h\left(\boldsymbol{\xi}_t, \boldsymbol{\xi}\right) \tag{135}$$

Now let's prove $\mathcal{A}$ is closed on $\mathbb{R}^d$:

(a) $\mathcal{A}\left(\boldsymbol{\xi}_t\right)$ is nonempty:
$\mathrm{E}_1\left(\boldsymbol{\xi}\right) = \frac{1}{2}\boldsymbol{\xi}^T\boldsymbol{\xi} + C_1$ is continuous, thus given any $\boldsymbol{\xi}_t$, $g\left(\boldsymbol{\xi}, \boldsymbol{\xi}_t\right)$ is continuous at every $\boldsymbol{\xi}$ in the compact set S according to its definition 51. By the Weierstrass theorem, $g\left(\boldsymbol{\xi}, \boldsymbol{\xi}_t\right)$ has minimum on S. Using 64, this minimum is global minimum on $\mathbb{R}^d$. Hence $\mathcal{A}\left(\boldsymbol{\xi}_t\right)$ is nonempty at any $\boldsymbol{\xi}_t$.

(b) $h\left(\mathbf{x}, \mathbf{y}\right)$ is continuous:
$\mathrm{E}_1\left(\boldsymbol{\xi}\right)$ is continuous. $\nabla_{\boldsymbol{\xi}} \mathrm{E}_2\left(\boldsymbol{\xi}\right) = \boldsymbol{X}\operatorname{softmax}\left(\beta\boldsymbol{X}^T\boldsymbol{\xi}\right)$ is continuous. Hence $g\left(\mathbf{x}, \mathbf{y}\right)$ is continuous at any $\mathbf{x}, \mathbf{y}$ according to its definition 51. Then $h\left(\mathbf{x}, \mathbf{y}\right)$ is continuous.

According to Lemma 12, $\mathcal{A}$ is closed on $\mathbb{R}^d$.

5. **Limit Points are Fixed Points**

Since

(a) All points $\boldsymbol{\xi}_{t+1}, t \geqslant 0$ are in the compact set S using 64.

(b) $\mathcal{A}$ is strictly monotonic with respect to E.

(c) $\mathcal{A}$ is closed on $\mathbb{R}^d$.

Using Lemma 14, all limit points of the sequence $\{\boldsymbol{\xi}_t\}_{t=0}^{\infty}$ are fixed points of 52.

6. **Limit Points are Stationary Points**

   Using Lemma 15, all fixed points of 52 are stationary points of 50.

7. **Adjacent Point Differences are Convergent**

   Using Lemma 16

$$\left\|\boldsymbol{\xi}_{t+1} - \boldsymbol{\xi}_t\right\|_2 \to 0, \text{ as } t \to \infty \tag{136}$$

8. **Limit Points are Compact**

   Using Lemma 17, the set of limit points of $\{\boldsymbol{\xi}_t\}_{t=0}^{\infty}$ is compact.

9. **Limit Points are Connected**

   Using Lemma 18, if $\{\boldsymbol{\xi}_t\}_{t=0}^{\infty}$ does not converge, then the set of limit points of $\{\boldsymbol{\xi}_t\}_{t=0}^{\infty}$ is connected.

10. **Convergence Sufficient Condition**

    Suppose $\{\boldsymbol{\xi}_t\}_{t=0}^{\infty}$ does not converge, then the set of limit points of $\{\boldsymbol{\xi}_t\}_{t=0}^{\infty}$ must contain at least two limit points, and is connected.

    Since $\mathcal{L}\left(\mathrm{E}^*\right)$ is finite, then the set of limit points can be decomposed into the union of two nonempty closed sets of points without common points. Hence it is not connected, which is a contradiction.

    Therefore, if $\mathcal{L}\left(\mathrm{E}^*\right)$ is finite, then any sequence $\{\boldsymbol{\xi}_t\}_{t=0}^{\infty}$ generated by the iteration 23 converges to some $\boldsymbol{\xi}^* \in \mathcal{L}\left(\mathrm{E}^*\right)$.

    $\square$

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